PRINCIPLES OF ANALYSIS
LECTURE 4 - DEDEKIND CUTS

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1. Example of Strong Induction

Let $x \in \mathbb{Z}$, $x \geq 2$. We say that x is *prime* if whenever $x = ab$ for some positive integers a and b, then either $a = 1$ or $b = 1$.

Problem 1. Let $x \in \mathbb{Z}$ be a positive integer, $x \geq 2$. Then x is the product of prime integers.

Proof. Proceed by induction on x, and select $x = 2$ as the base case. Clearly 2 is prime, and so it is the product of primes.

Now assume that every integer between 2 and $x - 1$ is a product of prime integers. If x is itself prime, we are done, so assume that x is not prime. Then there exist $a, b \in \mathbb{Z}$ such that $x = ab$ with $a \neq 1$ and $b \neq 1$. Then $a < x$ and $b < x$, so a is the product of primes and b is the product of primes. Therefore x is the product of primes. □

2. Jargon

Maximal and minimal (extremal) versus maximum and minimum (extremum).

Supremal and infimal versus supremum and infimum.

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3. Linearly Ordered Sets

A *linearly ordered set* $(A, \leq)$ is a set A together with a relation $\leq$ on A satisfying

(O1) $a \leq a$;
(O2) $a \leq b$ and $b \leq a$ implies $a = b$;
(O3) $a \leq b$ and $b \leq c$ implies $a \leq c$;
(O4) $a \leq b$ or $b \leq a$;

for every $a, b, c \in A$. We call $\leq$ a *linear order relation* on A.

Let A and B be linearly ordered sets. A *morphism* from A to B is a function $f : A \to B$ such that

$$a_1 \leq a_2 \Rightarrow f(a_1) \leq f(a_2).$$

Let A be a linearly ordered set. If $B \subset A$, the B naturally inherits the linear order, and becomes a linearly ordered set in its own right.

Let $b \in B$. We say that b is an *minimal element* of B if $b \leq c$ for every $c \in B$. Similarly, we say that b is a *maximal element* of B if $c \leq b$ for every $c \in B$.

We say that A is *dense* if for every $a_1, a_2 \in A$ with $a_1 < a_2$, there exists $a \in A$ such that $a_1 < a < a_2$.

Consider a partition $\{C, U\}$ of A into two blocks such that $c \leq u$ for every $c \in C$ and $u \in U$. There are four possibilities:

(a) C has a maximal element and U has a minimal element;
(b) C has a maximal element and U does not have a minimal element;
(c) C does not have a maximal element and U has a minimal element;
(d) C does not have a maximal element and U does not have a minimal element.

In cases **(a)**, **(c)**, and **(d)**, we say that C is a *cut*.
In case **(a)**, we say that C is a *jump*.
In case **(c)**, we say that C is a *hit*.
In case **(d)**, we say that C is a *gap*.

Observation 1. Let A be a linearly ordered set. Then A is dense if and only if A has no jumps.

Observation 2. The rational numbers $\mathbb{Q}$ is dense.

Proof. The average of two distinct rational numbers is rational and is between them. □

4. Dedekind Cuts

A *Dedekind cut* is a cut in the rational number line; that is, it is a proper nonempty subset $C \subset \mathbb{Q}$ such that

(C1) $c \in C$ and $u \in \mathbb{Q} \smallsetminus C$ implies $c < u$;
(C2) C has does not contain a maximal element.

The set of all Dedekind cuts is naturally ordered by inclusion. Moreover, this is a total order

5. Addition of Cuts

Let C_1 and C_2 be Dedekind cuts. Define their sum as

$$C_1 + C_2 = \{x \in \mathbb{Q} \mid x = c_1 + c_2 \text{ for some } c_1 \in C_1 \text{ and } c_2 \in C_2\}.$$

Proposition 1. *Let $C_1, C_2 \subset \mathbb{Q}$ be cuts. Then $C_1 + C_2$ is a cut.*

Proof. Set $C = C_1 + C_2$ and $U = \mathbb{Q} \smallsetminus C$. Clearly $C \subset \mathbb{Q}$ is nonempty; we wish to prove properties **(C1)** and **(C2)**.

Let $c \in C$ and $u \in U$. Then $c = c_1 + c_2$ for some $c_1 \in C_1$ and $c_2 \in C_2$. Suppose that $u \leq c$; then $u - c_2 \leq c_1$, which implies that $u - c_2 \in C_1$. Set $u - c_2 = a \in C_1$; then $u = a + c_2 \in C_1 + C_2 = C$, a contradiction. Thus $c < u$.

Since C_1 and C_2 are cuts, c_1 and c_2 are not maximal elements in C_1 and C_2, respectively. Thus there exists $a_1 \in C_1$ and $a_2 \in C_2$ such that $c_1 < a_1$ and $c_2 < a_2$. Then $a_1 + a_2 \in C$, and $c < a_1 + a_2$; thus c is not maximal in C. □

Let $M = \{x \in \mathbb{Q} \mid x < 0\}$. Clearly M is a Dedekind cut.

Let C be a Dedekind cut, and set

$$-C = \{x \in \mathbb{Q} \mid x = -y \text{ for some nonminimal } y \in \mathbb{Q} \smallsetminus C\}.$$

Proposition 2. *Let $C, C_1, C_2, C_3 \subset \mathbb{Q}$ be cuts. Then $-C$ is a Dedekind cut, and*

(F1) $(C_1 + C_2) + C_3 = C_1 + (C_2 + C_3)$;
(F2) $C + M = C$;
(F3) $C + (-C) = M$;
(F4) $C_1 + C_2 = C_2 + C_1$.

Proof. Exercise.

□

Define subtraction of Dedekind cuts in the usual way.

Let C be a Dedekind cut, and say that C is *positive* if M is a proper subset of C, and that C is *negative* if C is a proper subset of M.

6. Multiplication of Cuts

Let C_1 and C_2 be Dedekind cuts. Set

$$C_1 * C_2 = \{x \in \mathbb{Q} \mid x = c_1c_2 \text{ for some } c_1 \in C_1 \smallsetminus M \text{ and } c_2 \in C_2 \smallsetminus M\} \cup M.$$

Now define their product by

$$C_1 \cdot C_2 = \begin{cases} C_1 * C_2 & \text{if } C_1 \text{ and } C_2 \text{ are positive;} \\ -((-C_1) * C_2) & \text{if } C_1 \text{ is negative and } C_2 \text{ is positive;} \\ -(C_1 * (-C_2)) & \text{if } C_1 \text{ is positive and } C_2 \text{ is negative;} \\ (-C_1) * (-C_2) & \text{if } C_1 \text{ and } C_2 \text{ are negative;} \\ M & \text{if } C_1 = M \text{ for } C_2 = M. \end{cases}$$

Proposition 3. *Let $C_1, C_2 \subset \mathbb{Q}$ be cuts. Then $C_1 \cdot C_2$ is a cut.*

Proof. Again set $C = C_1 + C_2$ and $U = \mathbb{Q} \smallsetminus C$, and prove properties **(C1)** and **(C2)**. We assume that C_1 and C_2 are positive; the other cases require only minor adjustments.

Let $c \in C$ so that $c = c_1c_2$ for some $c_1 \in C_1 \smallsetminus M$ and $c_2 \in C_2 \smallsetminus M$; the other cases are easy.

Let $u \in U$; by definition, $M \subset C$ so $0 \leq u$. Suppose that $u \leq c$; then $u/c_2 \leq c_1$, which implies that $u/c_2 \in C_1$. Set $u/c_2 = a \in C_1$; then $u = ac_2 \in C_1 \cdot C_2 = C$, a contradiction. Thus $c < u$.

Since C_1 and C_2 are cuts, c_1 and c_2 are not maximal elements in C_1 and C_2, respectively. Thus there exists $a_1 \in C_1$ and $a_2 \in C_2$ such that $c_1 < a_1$ and $c_2 < a_2$. Then $a_1a_2 \in C$, and $c < a_1a_2$; thus c is not maximal in C. □

Let $I = \{x \in \mathbb{Q} \mid x < 1\}$. Clearly I is a Dedekind cut.

Let C be a Dedekind cut different from M, and set

$$C^{-1} = \{x \in \mathbb{Q} \mid x = y^{-1} \text{ for some } y \in C\}.$$

Proposition 4. *Let $C, C_1, C_2, C_3 \subset \mathbb{Q}$ be cuts. Then C^{-1} is a Dedekind cut, and*

- **(F5)** $(C_1 \cdot C_2) \cdot C_3 = C_1 \cdot (C_2 \cdot C_3)$;
- **(F6)** $C \cdot I = C$;
- **(F7)** $C \cdot (C^{-1}) = I$;
- **(F8)** $C_1 \cdot C_2 = C_2 \cdot C_1$;
- **(F9)** $C_1 \cdot (C_2 + C_3) = (C_1 \cdot C_2) + (C_1 \cdot C_3)$.

Proof. Exercise. □

7. Ordering of Cuts

Let $\mathcal{C} = \{C \subset \mathbb{Q} \mid C \text{ is a cut}\}$. Define a relation $\leq$ on $\mathcal{C}$ by

$$C_1 \leq C_2 \Leftrightarrow C_1 \subset C_2.$$

Proposition 5. *Let $C, C_1, C_2, C_3 \in \mathcal{C}$. Then*

(O1) $C \leq C$;
(O2) $C_1 \leq C_2$ *and* $C_2 \leq C_1$ *implies* $C_1 = C_2$;
(O3) $C_1 \leq C_2$ *and* $C_2 \leq C_3$ *implies* $C_1 \leq C_3$;
(O4) $C_1 \leq C_2$ *or* $C_2 \leq C_1$.

Moreover,

(O5) $C_1 \leq C_2$ *implies* $C_1 + C_3 \leq C_2 + C_3$;
(O6) $C_1 \leq C_2$ *implies* $C_1 \cdot C_3 \leq C_2 \leq C_3$ *whenever* $M \leq C_3$.

Proof. Exercise. □

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